Climate Dynamics
PartII: Atmospheric Dynamics

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This course consists of 8 lectures.

- At the end of each lesson, exercises will be given as homework and discussed in the beginning of the following lesson.
- *John A. Dutton*: The Ceaseless Wind. Dover Publications, INC., New York
  James R. Holton: Dynamic Meteorology, Third edition, Academic Press.
  Joseph Pedlosky: Geophysical Fluid Dynamics, Springer-Verlag.
  Others are suggested in the individual lectures. Many others are good as well, so choose!
- Lecture notes will be available at https://www.ictp.it/teaching-material/climate-dynamics-units-lecture-1, etc.
- If you find mistakes, corrections are highly appreciated!

Topics in the course

- Vorticity equation for synoptic-scale motion; potential vorticity conservation (barotropic and general) [1.5 h]
- Rossby waves; free Rossby waves; forced Rossby waves; turning latitude [1.5 h]
- Equatorial waves; Rossby-gravity waves; Kelvin waves [1.5 h]
- ENSO atmosphere and ocean feedback mechanisms; Gill model; Reduced Gravity Model [1.5h]

- Rainfall responses to heating; Ekman pumping effect; upper-level divergence [1.5h]

- The General Circulation; Hadley Cell; Ferrell Cell; Tropical zonal and meridional circulations; Walker circulation; Sverdrup balance [1.5h]

- Energetics of the General Circulation; Lorenz' energy cycle [1.5 h]

- Modes of variability in the climate system: ENSO, PDO, NAO, AMO [1.5h]

- Predictability of the Atmosphere, Lorenz Model; predictability measures [3h]

1 Vorticity equation for synoptic-scale motion

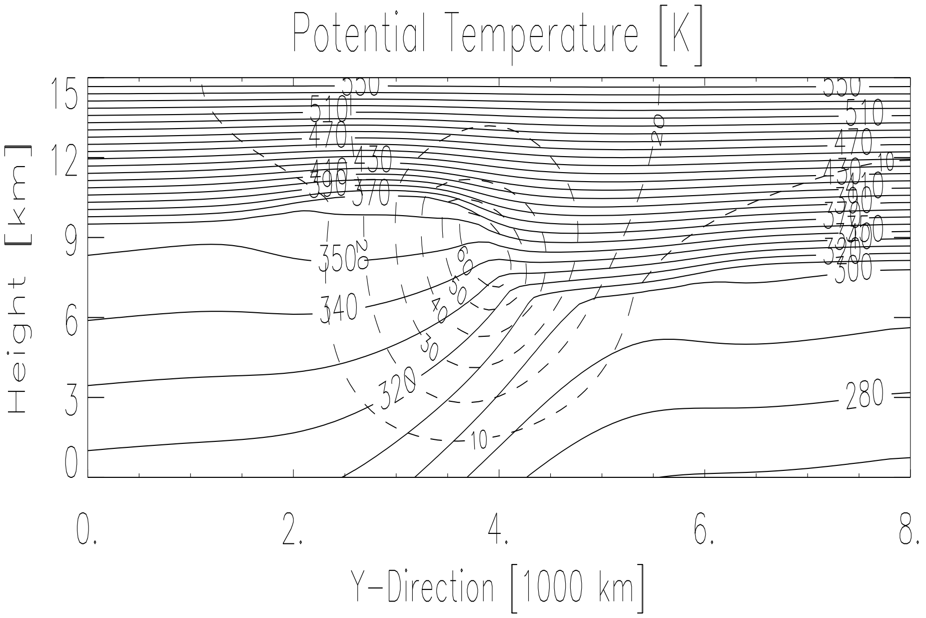

Figure 1: Idealised situation (meridional-vertical section) of the extratropical mean state. Potential temperature (solid lines, K) and zonal wind (dashed, m/s). As we will show later in ths Climate Dynamics course the wind approximately fulfill the thermal wind equation $\partial u_g/\partial z \approx -g/(fT)\ \partial T/\partial y$.

Vorticity is an important concept for the analysis of all kind of atmospheric motions, but in particular for large-scale atmospheric motions. We use the approximate horizontal equations of motion (in the vertical the equation of motion degenerates to the hydrostatic equation) on a sphere, but neglecting all metric terms that occur in the total derivative. Furthermore, we use the abbreviations $dx = rcos\phi\ d\lambda, dy = r\ d\phi, dz = dr$). Also recall the definition of the Coriolis parameter $f \equiv 2\Omega \sin\phi$, and note that we have already neglected the small term proportional to the vertical velocity Coriolis term.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad . \tag{2}$$

If we apply $\partial/\partial y$ to Eq. 1 and $\partial/\partial x$ to Eq. 2 and substract the first from the second, we obtain using the definition $\xi = \partial v/\partial x - \partial u/\partial y$ (Exercise!)

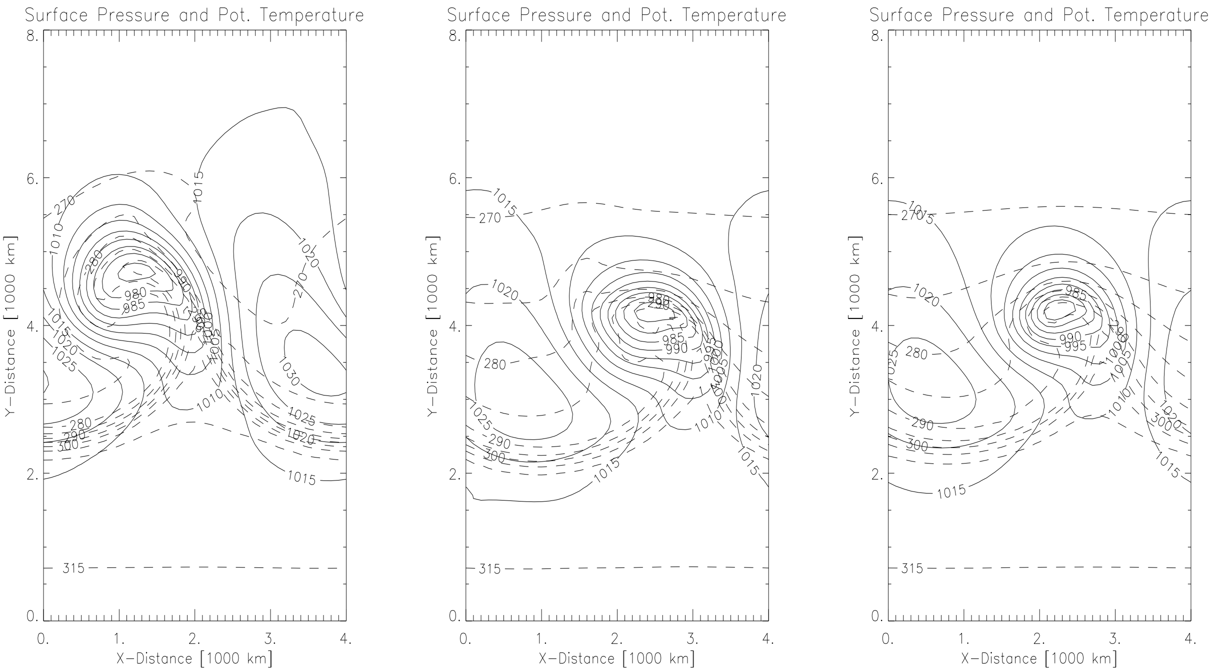

Figure 2: Typical surface pressure [hPa] and potential temperature [K] distributions in extratropical cyclones.

$$\begin{aligned} \frac{\partial \xi}{\partial t} \quad &+ \quad u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z} + (\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &+ \quad \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \end{aligned} \tag{3}$$

The coriolis parameter only depends on y, so we may write:

$$\begin{aligned} \frac{d}{dt}(\xi + f) = \quad &- \quad (\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ &- \quad \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \ . \end{aligned} \tag{4}$$

This is the vorticity equation usually used to analyse synoptic-scale motions (the direction is perpendicular to the earths surface). It states the the rate of change of absolute vorticity following the motion is given by the sum of three terms, called the divergence term, the tilting or twisting term, and the solenoidal term, respectively.

The first term on the right-hand side may be interpreted as expression of the angular momentum conservation. Imagine a ice skater who rotates and while rotating moves his arms closer to his body: His rotation accelerates (see Fig. 3). But since we are dealing with large scales, in Eq. 4 the absolute vorticity, $\eta = \xi + f$, has to be considered. The interpretation of the second term is that vertical vorticity may be generated by the tilting of horizontal vorticity components by a non-uniform vertical motion field. The meaning of the third term is the solenoidal term. It can be expressed as (exercise!):

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) = \frac{1}{\rho^2}(\nabla \rho \times \nabla p) \cdot \mathbf{k} \ , \tag{5}$$

where **k** is the unit vector in vertical direction. In order to create vorticity through the solenoidal term, lines of constant density have to intersect with lines of constant pressure. A land-sea breeze is a typical example where vorticity in created in such a way.

1.1 Scale analysis of the vorticity equation

In order to understand which terms and therefore mechanisms are the dominant ones in the vorticity equation (4) in this section a 'scale-analysis' will be performed. The scale analysis uses the dimensions of the synoptic scales we are interested in, but also observed magnitudes of flow velocities and other quantities. It is not a rigorous procedure (you use part of the answer as input), but it helps to identify dominant mechanisms.

The scales are given in the following table:

Table 1: Scale parameters for synoptic-scale flow.

U $\sim$ 10 m s^{-1}	horizontal velocity scale
W $\sim$ 1 cm s^{-1}	vertical velocity scale
L $\sim$ 10^6 m	length scale
H $\sim$ 10^4 m	vertical scale
$\delta p \sim$ 10 hPa	horizontal pressure scale
$\rho \sim$ 1 kg m^{-3}	mean density
$\delta\rho/\rho \sim 10^{-2}$	fractional density fluctuation
L/U $\sim$ 10^5 s	time scale
$f_0 \sim 10^{-4}$ s^{-1}	Coriolis parameter
$\beta = df/dy \sim 10^{-11}$ m^{-1} s^{-1}	'beta' parameter

This gives

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \sim \frac{U}{L} \sim 10^{-5} s^{-1} \quad , \tag{6}$$

and

$$\xi/f_0 \sim U/(f_0 L) \equiv Ro \sim 10^{-1} \quad , \tag{7}$$

the ratio of relative to planetary vorticity is equal to the *Rossby number*, which is small for synoptic flow. Therefore, ξ may be neglected compared to f in the divergence term in the vorticity equation

$$(\xi + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \approx f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{8}$$

The magnitudes of the various terms in equation 4 can now be estimated as follows:

$$\frac{\partial \xi}{\partial t}, u\frac{\partial \xi}{\partial x}, v\frac{\partial \xi}{\partial y} \quad \sim \quad \frac{U^2}{L^2} \sim 10^{-10} s^{-2}$$

$$
\begin{aligned}
w\frac{\partial \xi}{\partial z} &\sim \frac{WU}{HL} \sim 10^{-11}s^{-2} \\
v\frac{df}{dy} &\sim U\beta \sim 10^{-10}s^{-2} \\
f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) &\sim \frac{f_0 U}{L} \sim 10^{-9}s^{-2} \\
\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) &\sim \frac{WU}{HL} \sim 10^{-11}s^{-2} \\
\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) &\sim \frac{\delta\rho\delta p}{\rho^2 L^2} \sim 10^{-11}s^{-2}
\end{aligned}
$$

The estimation of the divergence is an overestimation. Indeed it will be shown later in this course the the divergent part of the flow (which is also the non-geostrophic part) is an order of magnitude smaller that the rotational part (geostrophic). We have therefore

$$
\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \sim 10^{-6}s^{-1} \quad , \tag{9}
$$

which means that the divergence is typically one order of magnitude smaller than the vorticity of synoptic-scale motion. Therefore,

$$
f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \sim 10^{-10}s^{-2} \quad . \tag{10}
$$

Therefore in the vorticity equation (4), we have the first order balance

$$
\frac{d_h(\xi+f)}{dt} = -f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \quad , \tag{11}
$$

where

$$
\frac{d_h}{dt} \equiv \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} \quad . \tag{12}
$$

The synoptic-scale vorticity equation 11 states that the rate of change of absolute vorticity following the horizontal motion is approximately given by the generation (destruction) of vorticity owing to horizontal convergence (divergence; see sketch 3). Indeed, this is considered to be the main mechanism of cyclone (and anticyclone) developments, connecting cyclonic motion to low pressure and anticyclonic motion to high pressure (only for large-scale motions!).

1.2 The Barotropic (Rossby) Potential Vorticity Equation

As barotropic model of the Atmosphere we assume that there is incompressibility and the flow may be confined by the height of two given boundaries, $h(x,y,z,t) = H_t - H_b$ (see also lecture on equatorial waves). The incompressibility condition may be expressed as

$$
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \quad . \tag{13}
$$

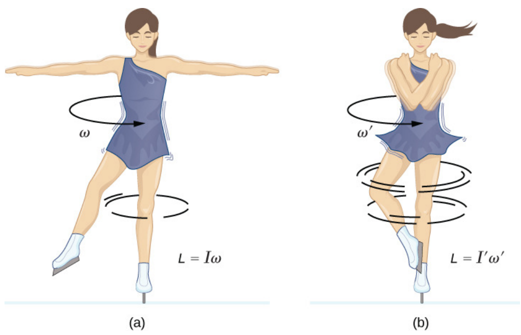

Figure 3: Sketch of ice scater closing in arms. From: courses.lumenlearning.com

We also assume that the horizontal velocities are independent of height. With this we can integrate the vorticity equation (11) vertically to obtain

$$h\frac{d_h(\xi+f)}{dt} = (f+\xi)[w(H_t) - w(H_b)] \quad . \tag{14}$$

Note that in equation (14), the relative vorticity ξ may be replaced, to a first approximation, by the geostrophic relative vorticity

$$\xi \approx \xi_g \equiv \nabla^2 gh/f_0 = \nabla^2\Phi/f_0 \quad , \tag{15}$$

where we have assumed that the meridional scale, L, is small compared to the radius of the earth so that the geostrophic wind may be defined using a constant reference latitude of the Coriolis parameter $f \approx f_0 \equiv 2\Omega\sin\phi_0$. Also, in the operator (12) the horizontal velocities may be approximated by the geostrophic ones

$$\mathbf{v} \approx \mathbf{v}_g \equiv f_0^{-1}\mathbf{k}\times\nabla gh = f_0^{-1}\mathbf{k}\times\nabla\Phi \quad , \tag{16}$$

Equations 15 and 16 can be derived from the geostrophic equations and integration of the hydrostatic equation for an incompressible fluid (exercise!). Note that, for beauty, we have re-introduced the small ξ effect on the rhs of Eq. (14). Since $w(H_t) = dH_t/dt, w(H_b) = dH_b/dt$ we have,

$$\frac{1}{\xi+f}\frac{d_h(\xi+f)}{dt} = \frac{1}{h}\frac{d_h h}{dt} \quad . \tag{17}$$

Integrating left and right side leads to

$$\frac{d_h}{dt}[\ln(\xi+f)] = \frac{d_h}{dt}[\ln h] \quad , \tag{18}$$

which implies that

$$\frac{d_h}{dt}\frac{(\xi+f)}{h} = 0 \quad , \tag{19}$$

which is the potential vorticity conservation theorem for a barotropic fluid, first obtained by C. G. Rossby. The quantity conserved following the horizontal motion is the barotropic potential vorticity. It explains nicely some features of the observed stationary waves, e.g. induced by the Rocky mountains. If the flow is purely horizontal, i.e. rigid lid and lower boundary, then we obtain the *barotropic vorticity equation*

$$\frac{d_h(\xi + f)}{dt} = 0 \quad , \tag{20}$$

which states that the absolute vorticity is conserved following the horizontal motion. The flow in the mid-troposhere approximately fulfils this condition and equation (20) my be used to explain the movement of air particles in Rossby waves!

Note that using the approximations (15) and (16) the barotropic vorticity equation (20) can be re-written in terms of the streamfunction $\psi \equiv \Phi/f_0 = gh/f_0$ or equivalently also in terms of geopotential $\Phi = gh$ (exercise!)

$$\frac{d_h}{dt}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0 \quad . \tag{21}$$

Also the operator d_h/dt can be expressed in terms of streamfunction

$$\frac{d_h}{dt} = \frac{\partial}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} \tag{22}$$

Equations (21) and (22) can be used conveniently to compute Rossby wave solutions numerically.

1.3 The exact potential vorticity conservation law; Ertel's potential vorticity

The barotropic potential vorticity conservation law 19 for is a very instructive special case (incompressible, barotropic fluid) of a much more general conservation law. Ertel (1942, Meteorologische Zeitung, **59**, 271-281) was the first to derive the law in the most general form. In order to derive it, we start from a general form of the equations of motion

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p - 2\rho\boldsymbol{\Omega}\times\mathbf{v} - \rho\nabla\phi - \nabla\cdot\mathbf{F} \quad , \tag{23}$$

where $\mathbf{F}$ is the frictional tensor, $\boldsymbol{\Omega}$ is the (constant) rotation vector of the earth. From Eq. 23 we may derive (exercise!) the full 3-dimentional vorticity equation, which is a generalization of Eq. 4

$$\frac{d\mathbf{vort_a}}{dt} = \mathbf{vort_a}\cdot\nabla\mathbf{v} - \mathbf{vort_a}\nabla\cdot\mathbf{v} + \frac{\nabla\rho\times\nabla p}{\rho^2} - \nabla\times\frac{\nabla\cdot\mathbf{F}}{\rho} \; , \tag{24}$$

where $\mathbf{vort_a} = 2\boldsymbol{\Omega} + \xi$ is the 3-dimensional absolute vorticity vector. Note, that whereas $\boldsymbol{\Omega}$ is constant, its components following the earth surface are not! Using the continuity equation, this may be re-written as (exercise!)

$$\frac{d}{dt}\frac{\mathbf{vort_a}}{\rho} = \frac{\mathbf{vort_a}}{\rho}\cdot\nabla\mathbf{v} + \frac{\nabla\rho\times\nabla p}{\rho^3} - \frac{1}{\rho}\nabla\times\frac{\nabla\cdot\mathbf{F}}{\rho} \quad . \tag{25}$$

Let us consider a scalar quantity, λ, which is a function of just ρ and p. For a one component system, any thermodynamic quantity could be used, but in the end it turns out that quantities that obey a conservation law are particularly useful. Therefore, the entropy is a very good candidate (or potential temperature in the atmosphere or potential density in the ocean). The equation may then look like

$$\frac{d\lambda}{dt} = \sigma_\lambda \quad . \tag{26}$$

Let us perform the following simple calculation

$$\mathbf{vort_a} \cdot \frac{d}{dt}\nabla\lambda = \mathbf{vort_a} \cdot \nabla\sigma_\lambda - [\mathbf{vort_a} \cdot \nabla\mathbf{v}] \cdot \nabla\lambda \quad . \tag{27}$$

On the other hand, if we multiply as scalar product Eq. 25 by $\nabla\lambda$, we get

$$\nabla\lambda \cdot \frac{d}{dt}\frac{\mathbf{vort_a}}{\rho} = \left[\frac{\mathbf{vort_a}}{\rho} \cdot \nabla\mathbf{v}\right] \cdot \nabla\lambda + \nabla\lambda \cdot \frac{\nabla\rho \times \nabla p}{\rho^3} - \frac{1}{\rho}\nabla\lambda \cdot \nabla \times \frac{\nabla \cdot \mathbf{F}}{\rho} \quad . \tag{28}$$

Combining Eqs. 27 and 28 leads to

$$\frac{d}{dt}\left[\frac{\nabla\lambda \cdot \mathbf{vort_a}}{\rho}\right] = \frac{1}{\rho^3}\nabla\lambda \cdot (\nabla\rho \times \nabla p) + \frac{\mathbf{vort_a}}{\rho} \cdot \nabla\sigma_\lambda - \frac{1}{\rho}\nabla\lambda \cdot \nabla \times \frac{\nabla \cdot \mathbf{F}}{\rho} \quad . \tag{29}$$

The quantity on the lhs of Eq. 29 is conserved following the motion if (why?, Exercise):

- The fluid is barotropic ($(\nabla\rho \times \nabla p = 0)$ or λ is a thermodynamic function of p and ρ, i.e. $\lambda = \lambda(p, \rho)$,
- the quantity λ is itself conserved following the motion, i.e. $\sigma_\lambda = 0$,
- the flow is frictionless ($\mathbf{F} = 0$).

In case the above criterion are fulfilled, we may call the quantity

$$q = \left[\frac{\nabla\lambda \cdot \mathbf{vort_a}}{\rho}\right] \tag{30}$$

potential vorticity. So far, apart from the momentum budget, no physical contrain is on the variable λ, which may be chosen as suitable. A good choice is the entropy s, which for adiabatic-reversible processes is a constant following the motion. For atmospheric purposes it is convenient to choose alternatively the potential temperature θ. Furthermore, for large-scale atmospheric (and ocean) dynamics it is generally a good approximation that the vorticity is dominated by its vertical component (note that this approximation may break down if there are strong horizontal potential temperature gradients). In this case the potential vorticity can be expressed as

$$q = \left[\frac{\eta}{\rho}\frac{\partial\theta}{\partial z}\right] \quad , \tag{31}$$

where $\eta = \xi + f$ is the vertical component of the absolute vorticity. The form 31 is most conveniently explored in the pressure coordinate system, which will be introduced in the next section. Note that a convenient and exact version of the potential vorticity can be derived using a coordinate system with the potential temperature as vertical coordinate.

The barotropic from of the potential vorticity Eq. 19 may be derived by considering the quantity

$$\lambda = \frac{z - H_b}{h} \quad , \tag{32}$$

which measures the relative height of a parcel with respect to the total height of the fluid and turns out to be conserved for barotropic flow with constant density.

Excercises

1. Show that applying $\partial/\partial y$ to Eq. 1 and $\partial/\partial x$ to Eq. 2 leads to Eq. 3.
2. Show the validity of Eq. 5.
3. Show that the geostrophic formulations 15 and 16 can be derived from the usual geostrophic wind

   $$\mathbf{v}_g \equiv \frac{1}{\rho f_0}\mathbf{k} \times \nabla p$$

   and that with this the barotropic vorticity equation 20 can be written as Eq. 21 with 22.
4. An air parcel at 30 N moves northward conserving absolute vorticity. If its initial relative vorticity is $5 \times 10^{-5} s^{-1}$, what is its relative vorticity upon reaching 90 N?
5. An air column at 60 N with $\xi = 0$ initially stretches from the surface to a fixed tropopause at 10 km. If the air column moves until it is over a mountain barrier 2.5 km high at 45 N, what are its absolute vorticity and relative vorticity as it passes the mountain top, assuming that the flow satisfies the barotropic potential vorticity equation?
6. Derive Eq. 24 from 23 and Eq. 25 from 24.
7. Discuss under which conditions 30 is conserved following the motion.

1.4 Recall: The basic equations in isobaric Coordinates

As you have derived last term, the governing thermo-hydrodynamics equations in pressure coordinates are:

Horizontal momentum equation:

$$\frac{d\mathbf{v}}{dt} + f\mathbf{k} \times \mathbf{v} = -\nabla\Phi \quad . \tag{33}$$

The total derivative d/dt is:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{34}$$

$\omega = dp/dt$ (called the 'omega' vertical velocity) is the pressure change following the motion. Note that when w is positive ω is typically negative.

Hydrostatic equation:

$$\frac{\partial\Phi}{\partial p} = -\frac{1}{\rho} = -\alpha = -\frac{RT}{p} \quad . \tag{35}$$

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \ . \tag{36}$$

In pressure coordinates the full continuity equation takes the form of that of an incompressible fluid, i.e. the time derivative of density does not occur anymore explicitely.

Thermodynamic energy equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{Q}{cp} \quad , \tag{37}$$

where the stability factor

$$S_p = \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \tag{38}$$

has been introduced. In Eq. (38) we have used the definition of the potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \quad . \tag{39}$$

p_0 is a constant reference pressure here. Using the dry adiabatic lapse rate $\Gamma_d = g/cp$,

$$S_p = (\Gamma_d - \Gamma)/(\rho g) \quad , \tag{40}$$

where the definition of the lapse rate $-dT/dz = \Gamma$ has been used.

For some purposes the diabatic heating may be just the condensational heating $Q = -L_{lv}\frac{dm_v}{dt}$

Exercises

1. Show that
$$\frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} = (\Gamma_d - \Gamma)/(\rho g)$$
using the definition of potential temperature, and dry adiabatic and actual lapse rates.