5 Equatorial Wave Theory

Lecture based on 'An Introduction to Dynamic Meteorology' by J. R. Holton, *Academic Press, INC.*, 3rd edition, 511 pp.

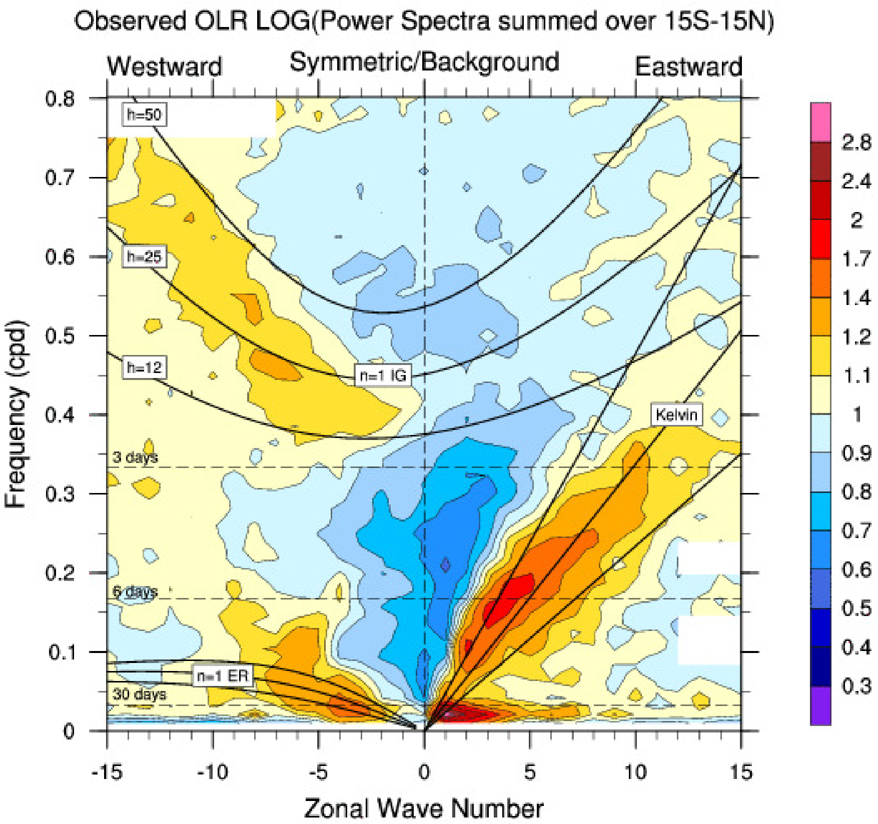

Figure 11: Dispersion diagram for tropical Outgoing longwave radiation. Source: www.cgd.ucar.edu,NOAA.

Equatorial waves are an important class of eastward and westward propagating disturbances that are trapped about the equator (that is, they decay away from the equatorial region). In a dispersion diagram for observed equatorial quantities, these wave can be identified as regions of increased energy density (Fig. 11).

Diabatic heating by organized tropical convection may excite equatorial wave motions (see Fig. 12). Through such waves the dynamical effects of convective storms can be communicated over large longitudinal distances in the tropics. Such waves can produce remote responses to localized heat sources. Furthermore, by influencing the pattern of low-level moisture convergence they can partly control the spatial and temporal distribution of convective heating. In order to introduce equatorial waves in the simplest possible context, we here use a shallow-water model

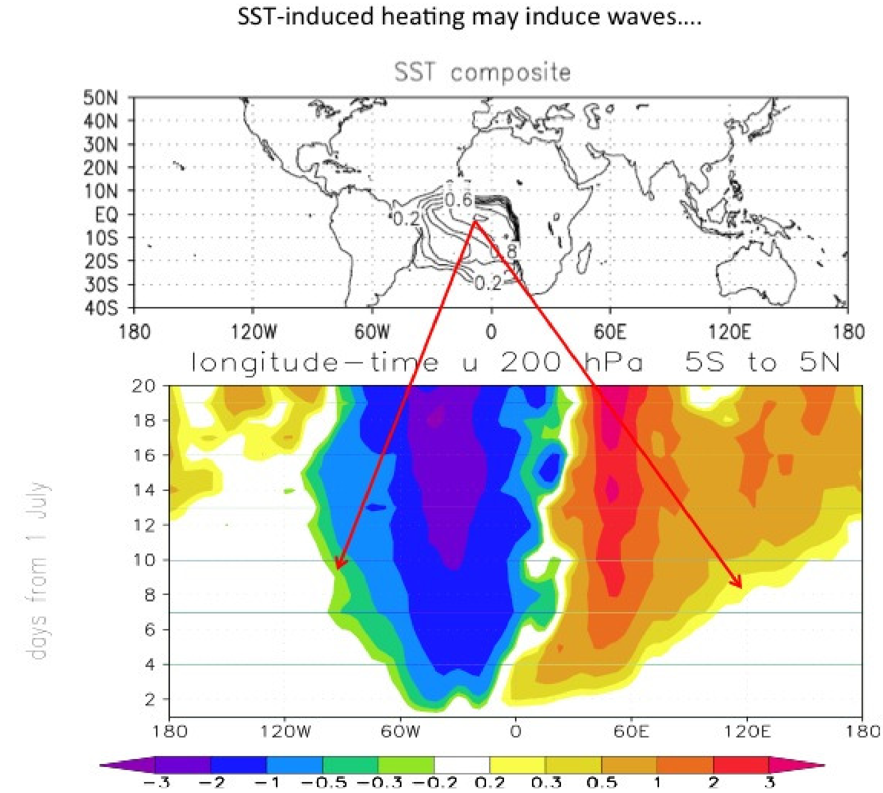

Figure 12: Equatorial Kelvin and Rossby waves triggered by an SST-induced heating. Source: Kucharski et al. 2008: A Gill-Matsuno-type mechanism explains the tropical Atlantic influence on African and Indian monsoon rainfall. Q. J. R. Meteorol. Soc. (2009), 135, 569-579, DOI: 10.1002/qj.406

and concentrate on the *horizontal* structure.

5.1 The shallow water equations

The shallow water equations are a drastic simplification to the real atmospheric flow. However, despite it's simplicity it gives often a good insight into many atmospheric wave phenomena. The basic assumptions in the shallow water model are

(i) The flow is incompressible $\rho = const.$

(ii) The flow is shallow enough so that the horizontal velocity components are independent of height.

(iii) The flow is hydrostatic. Accelerations in the vertical direction may be neglected.

Let us consider the horizontal momentum equations 1 and the hydrostatic equation 34

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv && (132)\\
\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu && (133)\\
\frac{1}{\rho}\frac{\partial p}{\partial z} &= -g\ , && (134)
\end{aligned}
$$

Further consider the continuity equation

$$
\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\ . \qquad (135)
$$

Integrating the hydrostatic equation from a height z to the top of the fluid leads to (assuming the pressure is vanishing there)

$$
\begin{aligned}
\int_z^{h(x,y,t)} \frac{\partial p}{\partial z}dz &= -\int_z^{h(x,y,t)} \rho g dz\ , \quad \text{or} && (136)\\
-p(x,y,z,t) &= -\rho g[h(x,y,t) - z]\ . && (137)
\end{aligned}
$$

Thus the horizontal pressure gradient force in the equations of motion 132, 133 may be expressed as

$$
\begin{aligned}
-\frac{1}{\rho}\frac{\partial p}{\partial x} &= -g\frac{\partial h}{\partial x} = -\frac{\partial \Phi}{\partial x} && (138)\\
-\frac{1}{\rho}\frac{\partial p}{\partial y} &= -g\frac{\partial h}{\partial y} = -\frac{\partial \Phi}{\partial y}\ , && (139)
\end{aligned}
$$

where we have defined $\Phi(x,y,t) = gh(x,y,t)$. Thus, keeping in mind that there the horizontal velocities do not depend on the vertical direction and ignoring the coriolis term proportional to the vertical velocity, the horizontal equations of motion may be written as

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u &= -\frac{\partial \Phi}{\partial x} + fv && (140)\\
\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v &= -\frac{\partial \Phi}{\partial y} - fu\ , && (141)
\end{aligned}
$$

The number of dependent variables in Eqs. 140 and 141 is reduced to 3, (u, v, Φ). Thus, if we have another equation only containing (u, v, Φ), then the system may be complete. This is achieved by simplification of the continuity equation 135 and vertical integration. First, we note that because of $\rho = const$, Eq. 135 reduces to

$$
\frac{\partial w}{\partial z} = -\nabla\cdot\mathbf{v}\ . \qquad (142)
$$

If we integrate this equation vertically from 0 to $h(x,y,t)$ we have

$$\int_0^h \frac{\partial w}{\partial z}\, dz = -\int_0^h \nabla \cdot \mathbf{v}\, dz \tag{143}$$

$$w(h) := \frac{dh}{dt} = \frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h = -(\nabla \cdot \mathbf{v})h \tag{144}$$

Eq. 144 may as well be written as

$$\frac{\partial \Phi}{\partial t} + \mathbf{v} \cdot \nabla \Phi = -\Phi \nabla \cdot \mathbf{v} \ . \tag{145}$$

Eqs. 140, 141 and 145 build a complete set of differential equations for (u, v, Φ), and are called the *shallow water equations.*

5.2 Linearization for an Equatorial β-plane

Now we linearize the set of equations 140, 141 and 145 about a motionless mean state with height h_e on an equatorial β-plane. Generally speaking, the β-plane assumption states that $f = 2|\omega| \sin\phi \approx f_0 + \beta y$, that is the $\sin\phi$-dependence is approximated linearly for a given latitude ϕ_0 by a Taylor series expansion (therefore $\beta = 2|\omega| cos\phi_0/\mathrm{a}$; a being the mean radius of the earth). If we set the base point at the equator we have $f_0 = 0$, therefore $f \approx \beta y$.

$$\frac{\partial u'}{\partial t} = -\frac{\partial \Phi'}{\partial x} + \beta y v' \tag{146}$$

$$\frac{\partial v'}{\partial t} = -\frac{\partial \Phi'}{\partial y} - \beta y u' \tag{147}$$

$$\frac{\partial \Phi'}{\partial t} = -g h_e \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \ , \tag{148}$$

where the primed variables denote the perturbations from the basic state. This is our basic set of linearized equations (with variable coefficients!) to study equatorial wave dynamics. By adjusting the scale height h_e as well the ocean case may be included.

Discuss Inertia-Gravity waves for extratropical situation and approximation $f = f_0 = const$, and assume $u'(x,t), v'(x,t), \Phi'(x,t)$.

5.2.1 Equatorial Rossby and Rossby-Gravity Modes

In order to find solutions to the linearized system 146, 147 and 148, we assume that the y-dependence can be separated

$$\begin{pmatrix} u' \\ v' \\ \Phi' \end{pmatrix} = \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\Phi}(y) \end{bmatrix} e^{i(kx - \nu t)} \qquad . \tag{149}$$

Substitution of Eq. 149 into 146-148 then yields a set of ordinary differential equations in y for the meridional structure functions $\hat{u}, \hat{v}, \hat{\Phi}$:

$$
\begin{aligned}
-i\nu\hat{u} &= -ik\hat{\Phi} + \beta y\hat{v} && (150)\\
-i\nu\hat{v} &= -\frac{\partial\hat{\Phi}}{\partial y} - \beta y\hat{u} && (151)\\
-i\nu\hat{\Phi} &= -gh_e\left(ik\hat{u} + \frac{\partial\hat{v}}{\partial y}\right) . && (152)
\end{aligned}
$$

If Eq. 150 is solved for $\hat{u} = k/\nu\hat{\Phi} + i\beta y\hat{v}/\nu$ and inserted into Eq. 151 and 152 we obtain

$$
\begin{aligned}
(\beta^2 y^2 - \nu^2)\hat{v} &= ik\beta y\hat{\Phi} + i\nu\frac{\partial\hat{\Phi}}{\partial y} && (153)\\
(\nu^2 - gh_e k^2)\hat{\Phi} + i\nu gh_e\left(\frac{\partial\hat{v}}{\partial y} - \frac{k}{\nu}\beta y\hat{v}\right) &= 0 . && (154)
\end{aligned}
$$

Finally, Eq. 154 is inserted into Eq. 153 to eliminate $\hat{\Phi}$, yielding a second-order differential equation in the single unknown, $\hat{v}$

$$
\frac{\partial^2\hat{v}}{\partial y^2} + \left[\left(\frac{\nu^2}{gh_e} - k^2 - \frac{k}{\nu}\beta\right) - \frac{\beta^2 y^2}{gh_e}\right]\hat{v} = 0 . \qquad (155)
$$

We seek solutions of this equation for the meridional distribution of $\hat{v}$, subject to the boundary condition that the disturbance fields vanish for $|y| \to \infty$. This boundary condition is necessary since the approximation $f \approx \beta y$ is nor valid for latitudes much beyond $\pm 30°$, so that solutions must be equatorially trapped if they are to be good approximations to the exact solutions on the sphere. Equation 155 differs from the classic equation for a harmonic oscillator in y because the coefficient in square brackets is not a constant but is a function of y. For sufficiently small y this coefficient is positive and solutions oscillate in y, while for large y, solutions either grow or decay in y. Only the decaying solutions, however, can satisfy the boundary conditions.

It turns out that solutions to Eq. 155 which satisfy the condition of decay far from the equator exist only when the constant part of the coefficient in square brackets satisfies the relationship (which is as well the dispersion relation!)

$$
\frac{\sqrt{gh_e}}{\beta}\left(-\frac{k}{\nu}\beta - k^2 + \frac{\nu^2}{gh_e}\right) = 2n + 1; \quad n = 0, 1, 2, ... \qquad (156)
$$

which is a cubic dispersion equation determining the frequencies of permitted equatorially trapped free oscillations for zonal wave number k and meridional mode number n. These solutions can be expressed most conveniently if y is replaced by the nondimensional meridional coordinate

$$
\xi = \left(\frac{\beta}{\sqrt{gh_e}}\right)^{1/2} y . \qquad (157)
$$

With the Eqs. 156 and 157, Eq. 155 becomes

$$\frac{\partial^2 \hat{v}}{\partial \xi^2} + \left(2n + 1 - \xi^2\right) \hat{v} = 0 \ . \tag{158}$$

This is the differential equation for a quantum mechanical, simple harmonic oscillator. The solution has the form

$$\hat{v}(\xi) = H_n(\xi) e^{-\xi^2/2} \ , \tag{159}$$

where $H_n(\xi)$ designates the *nth Hermite polynomial*. The first of these polynomials have the values

$$H_0 = 1, \quad H_1(\xi) = 2\xi, \quad H_2(\xi) = 4\xi^2 - 2 \ . \tag{160}$$

Thus, the index n corresponds to the number of nodes in the meridional velocity profile in the domain $|y| < \infty$. Inserting the solution 159 into Eq. 158 leads to one of the defining differential equations for Hermite polynomials. In general, the three solutions of Eq. 156 can be interpreted as eastward- and westward-moving equatorially trapped gravity waves and westward-moving equatorial Rossby waves. The case $n = 0$ (for which the meridional velocity perturbation has a gaussian distribution centered at the equator) must be treated separately. In this case the *dispersion relationship* 156 (which is something like a characteristic equation that gives the $\nu(k)$-dependence from which we may derive the phase velocities) factors as

$$\left(\frac{\nu}{\sqrt{gh_e}} - \frac{\beta}{\nu} - k\right)\left(\frac{\nu}{\sqrt{gh_e}} + k\right) = 0 \ . \tag{161}$$

The root $\nu/k = -\sqrt{gh_e}$, corresponding to a westward-propagating gravity wave, is not permitted since the second term in parentheses in Eq. 161 was explicitly assumed not to vanish when Eqs. 153 and 154 were combined to eliminate Φ. The roots given by the first term in parentheses in Eq. 161 are

$$\nu = k\sqrt{gh_e}\left[\frac{1}{2} \pm \frac{1}{2}\left(1 + \frac{4\beta}{k^2\sqrt{gh_e}}\right)^{1/2}\right] \ . \tag{162}$$

The positive root corresponds to an eastward-propagating equatorial inertio-gravity wave, while the negative root corresponds to a westward-propagating wave, which resembles an inertio-gravity wave for long zonal scale $k \to 0$ and resembles a Rossby wave for zonal scales characteristic of synoptic-scale disturbances. This mode is generally referred to as a Rossby-gravity wave.

5.2.2 Equatorial Kelvin Waves

In addition to the modes discussed in the previous section, there is another equatorial wave that is of great practical importance. For this mode, which is called the equatorial *Kelvin wave*, the meridional velocity perturbation vanishes and Eqs. 150 to 152 are reduced to the simpler set

$$
\begin{aligned}
-i\nu\hat{u} &= -ik\hat{\Phi} && (163)\\
\beta y\hat{u} &= -\frac{\partial\hat{\Phi}}{\partial y} && (164)\\
-i\nu\hat{\Phi} &= -gh_e\,(ik\hat{u})\ . && (165)
\end{aligned}
$$

Eliminating Φ between Eq. 163 and Eq. 165, we see that the Kelvin wave dispersion equation is that of the shallow-water gravity wave

$$
c^2 = \left(\frac{\nu}{k}\right)^2 = gh_e\ . \tag{166}
$$

According to Eq. 166 the phase speed c can be either positive or negative. But, if Eq. 163 and Eq. 164 are combined to eliminate Φ we obtain a first-order equation for determining the meridional structure

$$
\beta y\hat{u} = -c\frac{\partial\hat{u}}{\partial y}\ , \tag{167}
$$

which may be integrated immediately to yield

$$
\hat{u} = u_0\ e^{-\beta y^2/(2c)}\ , \tag{168}
$$

where u_0 is the amplitude of the perturbation zonal velocity at the equator. Equation 168 shows that if solutions decaying away from the equator are to exist, the phase speed must be positive ($c > 0$). Thus Kelvin waves are eastward propagating and have zonal velocity and geopotential perturbations that vary in latitude as Gaussian functions centered on the equator. The e-folding decay width is given by

$$
Y_K = |2c/\beta|^{1/2}\ , \tag{169}
$$

which for a phase speed $c = 30$ m s^{-1} gives $Y_K = 1600$ km. The meridional force balance for the Kelvin mode is an exact geostrophic balance between the zonal velocity and the meridional pressure gradient. It is the change in sign of the Coriolis parameter at the equator that permits this special type of equatorial mode to exist.

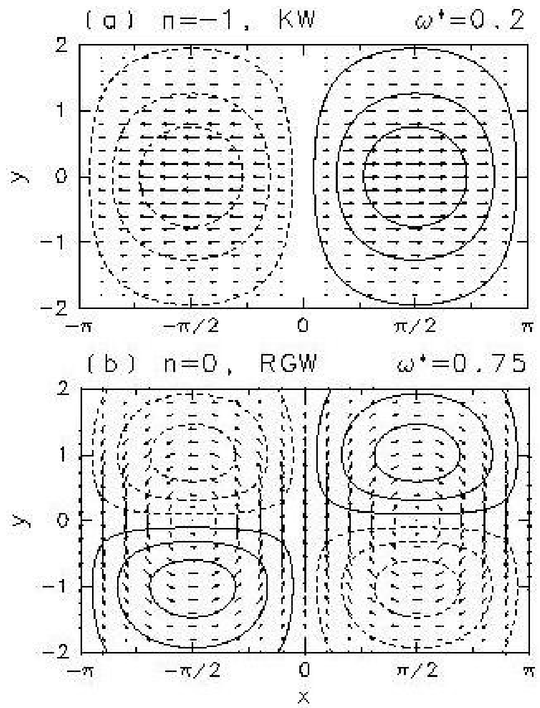

Fig.1 Horizontal structure of (a) Kelvin wave and (b) mixed-Rossby gravity wave. Contours show geopotential height component and arrows show horizontal wind components.

Figure 13: Illustration of Kelvin (upper panel) and Rossby-gravity (lower panel) waves.

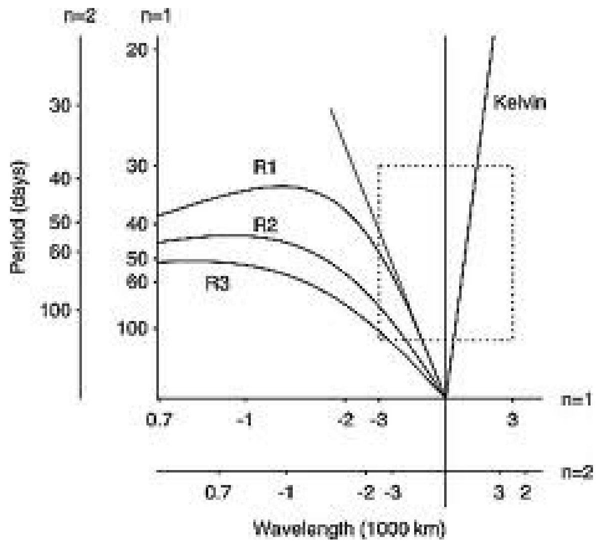

Figure 14: Dispersion diagram for equatorial Rossby-gravity and Kelvin waves.