11 Energetics of the general circulation

Suggested textbooks, reading:

a) Lorenz, E. N.; 1955: Available Potential Energy and the maintainence of the General Circulation. *Tellus*, **7**, 157-167

b) Tailleux, R.; 2013: Available Energy and Exergy in Stratified Fluids. *Annual Review of Fluid Mechanics*, **45**, 35-58, doi: 10.1146/annurev-fluid-011212-14620

c) Marquet, M.; 1991: On the concept of Exergy and availabale enthalpy: application to atmospheric energetics. *Q. J. R. Meteorolog. Soc.*, **117**, 449-475

d) Kucharski, F.; 1997: On the concept of exergy and available potential energy. *Q. J. R. Meteorolog. Soc.*, **123**, 2141-2156

We go back to the full (unapproximated) momentum equation 23

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - 2\rho \boldsymbol{\Omega} \times \mathbf{v} - \rho \nabla \phi - \nabla \cdot \mathbf{F} \quad , \tag{246}$$

where $\mathbf{F}$ is the frictional tensor. Multiplying with the velocity gives

$$\rho \frac{dk}{dt} = -\nabla \cdot (p\mathbf{v} + \mathbf{F} \cdot \mathbf{v}) + p\nabla \cdot \mathbf{v} - \rho \mathbf{v} \cdot \nabla \phi + \mathbf{F} : \nabla \mathbf{v} \quad , \tag{247}$$

where $k = 1/2(\mathbf{v} \cdot \mathbf{v})$ is the kinetic energy. We can reformulate equation 247 as equation

$$\rho \frac{d(k + \phi)}{dt} = -\nabla \cdot (p\mathbf{v} + \mathbf{F} \cdot \mathbf{v}) + p\nabla \cdot \mathbf{v} + \mathbf{F} : \nabla \mathbf{v} \quad . \tag{248}$$

The equation for internal energy is

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J} - p\nabla \cdot \mathbf{v} - \mathbf{F} : \nabla \mathbf{v} \quad . \tag{249}$$

$\mathbf{J}$ can contain the radiative flux vector and the diffusive (molecular) heat flux and these only heating terms for a one-component system in which no phase transitions are possible. The addition of condensational heating is an approximation to a one-component system and strictly possible only if more components are considered (dry air, water vapour and liquid water). The pressure work term $p\nabla \cdot \mathbf{v}$ may therefore be interpreted as reversible conversion term between kinetic plus potential and internal energy. The dissipational heating, $\mathbf{F} : \nabla \mathbf{v}$, is just transferring energy into the internal energy reservoir. The other terms are energy fluxes into the climate system. If we for the time being assume a closed system, then these terms vanish after an integration over the this system. In this case, we may assume that reservoirs of

internal, potential and kinetic energy can only change by exchanging energy between them.

Imagine an initial situation with a given temperature distribution, but without motion (this is, for example, the typical initial condition of ICTP AGCM). How much kinetic energy could the climate 'gain' in such a situation? A naive estimation would be $u = c_v T \approx 2 \cdot 10^5$ J/kg as typical local value. However, experience tells us that typical values of specific kinetic energy are about 10^2 J/kg, so that there is an overestimation of a factor of about 1000 in this simple (an naive) approach. Clearly, only a small portion of the large internal energy reservoir can be released into kinetic energy. It is obvious that we have to consider *differences* with respect to some reference state in order to define the proper amount of internal energy that is available for conversion into kinetic energy. An approach like $\Delta u = c_v \Delta T$, where $\Delta T = T - T_r$, with T_r some reference temperature, leads still to overestimations, but has the even more severe problem of not being positively definite. Note that the above arguments also apply to the potential energy, because of the proportionality of their total amounts in the atmophere ($P = R/c_v U$; Exercise!). The problem of identifying the energy available for conversion into kinetic energy is a classical one. For the atmosphere, Lorenz (1955) has developed a new energy concept, called *Available Potential Energy*. We can re-write the equations in a more suitable form:

$$\rho \frac{dk}{dt} = -\nabla \cdot (\mathbf{F} \cdot \mathbf{v}) - \mathbf{v} \cdot (\nabla p + \rho \nabla \phi) + \mathbf{F} : \nabla \mathbf{v} \quad . \tag{250}$$

$$\rho \frac{d(u + \phi)}{dt} = -\nabla \cdot (p\mathbf{v}) + \mathbf{v} \cdot (\nabla p + \rho \nabla \phi) + T\rho \frac{ds}{dt} \quad . \tag{251}$$

s is the specific entropy, governing all irreversible processes (for an ideal gas we have $s = c_p ln\theta$). From these equations we see that in a hydrostatic atmosphere ($\nabla p = -\rho \nabla \phi$), there is no reversible conversion from the internal plus potential to kinetic energy.

11.1 On Dry and Moist Static Energy

Before we move on to discuss the problem of the available potential energy, it is useful to derive from these exact equations the (approximate) conservation law for Dry Static Energy $= h + \phi$ (DSE; Eq. 132 from the lecture course on Earth System Thermodynamics). Please note that

$$-\nabla \cdot (p\mathbf{v}) = -\rho \frac{d}{dt}(p/\rho) + \frac{\partial p}{\partial t}. \tag{252}$$

With this adding Eq. (250) and Eq. (251) we get

$$\rho \frac{d(k + u + p/\rho + \phi)}{dt} = \rho \frac{d(k + h + \phi)}{dt} = \frac{\partial p}{\partial t} - \nabla \cdot (\mathbf{F} \cdot \mathbf{v}) + \mathbf{F} : \nabla \mathbf{v} + T\rho \frac{ds}{dt} \quad . \tag{253}$$

This demonstrates that the DSE is approximately conserved for adiabatic-reversible flow, for which kinetic energy changes are small compared to changes of enthalpy and

potential energy, and for which the local pressure tendency is also sufficiently small. These last approximations go along with the hydrostatic approximation which has been made when deriving the dry static energy conservation (Eq. 132 of EST lecture course).

If we consider a system with moisture, we may approximate it's effect on the thermodynamic energy equation 249 according to 171 by adding the term $-L_{lv}\frac{dm_v}{dt}$ on the rhs of Eq. 253. This, using the good approximation of $L_{lv} = const.$ leads to the moist version of Eq. 253, *Moist Static Energy* $= h + \phi + L_{lv}m_v$ (MSE) equation:

$$\rho\frac{d(k + h + \phi + L_{lv}m_v)}{dt} = \frac{\partial p}{\partial t} - \nabla \cdot (\mathbf{F} \cdot \mathbf{v}) + \mathbf{F} : \nabla\mathbf{v} + T\rho\frac{ds}{dt} \quad . \tag{254}$$

The conservation of MSE requires the same approximate conditions as the conservation of the DSE.

Let's go back to derive the available potential energy. If in Eq. 251 we define a function that just depends on entropy, $T_0(s)$, then it is possible to remove from the internal plus potential energy a part that merely depends on entropy:

$$\rho\frac{d(u + \phi - q_0(s))}{dt} = -\nabla \cdot (p\mathbf{v}) + \mathbf{v} \cdot (\nabla p + \rho\nabla\phi) + (T - T_0(s))\rho\frac{ds}{dt} \quad , \tag{255}$$

where $q_0(s) = \int_{s_B}^{s} T_0(s')ds'$. Note that for isothermal processes, $T_0(s) = T$, then $q_0(s) = Ts + const.$ and $u + \phi - q_0(s) = u - Ts + \phi$, and $(u - Ts)$ is the *Free Energy* (a famous quantity for available energy for isothermal processes). The interpretation is that we can remove from the internal plus potential energy amounts a portion that is not available for conversion into kinetic energy. However, the problem of the positive definiteness of the *useful* energy $u + \phi - q_0(s)$ remains. We have to use a transformation of thermodynamic variables that guarantees positive definiteness in the end.

11.2 Exergy transformation

The exergy transformation is usually applied to the internal energy u to provide a positive quantity with properties of a thermodynamic potential (see Figure 52).

Application of this transformation to the energy $u + \phi - q_0$ results in

$$e_{ape} = \Delta u - \Delta q_0 + p_R\Delta\alpha \quad , \tag{256}$$

where $\Delta\psi = \psi - \psi_R$, where the subscript R indicates the reference state that depends only on z. The quantity e_{ape} is the available potential energy first introduced by Lorenz (1955) in a globally integrated form. It is a positive quantity the for a closed volume and the only *reversible* production/destruction term is the conversion to/from kinetic energy. This quantity therefore has all the required properties. The reference state is arbitrary at this point, but following our idea of e_{ape} being the available energy, it should be derived from a variational principle, minimizing the volume integral of e_{ape}. However, for practical purposes, determination of the
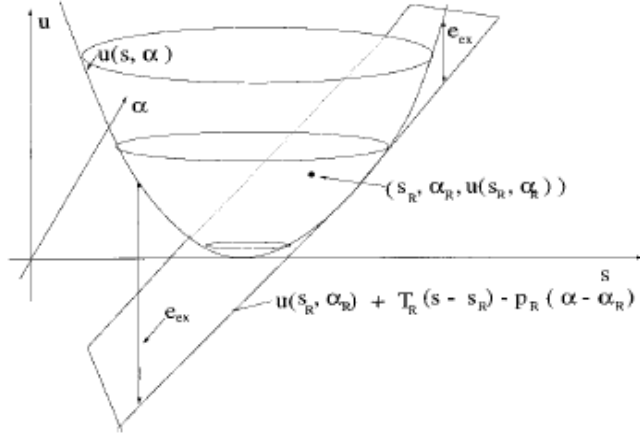
Figure 52: The convex internal energy, u, as a function of specific entropy, s, and specific volume, α, in phase space. The exergy transformation is just the (always positive) difference between u the tangente at the reference point.

reference state by, for example, horizontal averaging of temperature is sufficient. All other reference state variables can be determined by vertically integrating the hydrostatic equation for an ideal gas. Note that e_{ape} according to Eq. 256 becomes the classical exergy in case of an isothermal reference temperature $T_0(s) = const. = T_R$. Finally, for an ideal gas, the lowest order approximation of e_{ape} is

$$e_{ape} \approx \frac{1}{2} \left[\frac{g^2}{N_R^2} \left(\frac{\Delta\theta}{\theta_R} \right)^2 + RT_R \frac{c_v}{c_p} \left(\frac{\Delta p}{p_R} \right)^2 \right] \quad , \tag{257}$$

where

$$N_R^2 := \frac{g}{c_p} \frac{ds_R}{dz} = \frac{g^2}{c_p^2} \left[\frac{T_R}{c_p} - \frac{dT_R}{dz} \left(\frac{ds_R}{dz} \right)^{-1} \right]^{-1} \quad . \tag{258}$$

From Eq. 258 follows that e_{ape} is positive if the stratification of the reference state is statically stable ($N_R^2 > 0$). For application of this concept to the general circulation of the atmosphere this condition is certainly fulfilled (just look at a potential temperature cross section in meridional and height direction, e.g. Fig. 3). However, to a system where the stratification is unstable even in the horizontal average, the available potential energy concept cannot be applied, and we have to return to other methods of identifying useful energy. The balance equation for e_{ape} can be derived by noting that

$$\frac{de_{ape}}{dt} = (T - T_0(s)) \frac{ds}{dt} - \Delta p \frac{d\alpha}{dt} + \Delta\alpha \frac{dp_R}{dt} \tag{259}$$

The energy equations 250 and 251 can be reformulated in the following form

$$\rho \frac{dk}{dt} = -\nabla \cdot (\Delta p \mathbf{v} + \mathbf{F} \cdot \mathbf{v}) + \Delta p \nabla \cdot \mathbf{v} + \rho \frac{\Delta\alpha}{\alpha_R} \mathbf{v} \cdot \nabla\phi + \mathbf{F} : \nabla \mathbf{v} \tag{260}$$

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$$\rho \frac{de_{ape}}{dt} \quad = \quad -\Delta p \nabla \cdot \mathbf{v} - \rho \frac{\Delta \alpha}{\alpha_R} \mathbf{v} \cdot \nabla \phi + \rho \frac{T - T_o(s)}{T} q \quad , \tag{261}$$

where the abbreviation for diabatic processes $T ds/dt = q$ has been used. The factor $(T - T_0(s))/T = \eta$ may be interpreted as *Carnot factor* controlling the efficiency of the energy gain for a given heating. It can be approximated to the first order as (exercise!)

$$\eta \approx \left(\frac{g^2}{c_p N_R^2} \frac{\Delta \theta}{\theta_R} + T_R \frac{R}{c_p} \frac{\Delta p}{p_R} \right) \frac{1}{T} \quad . \tag{262}$$

The interpretation is then that the volume integrated generation of e_{ape} is dependent of the correlation of η and q. It's not the heating per se that is important, but the differential heating has to be correlated with potential temperature perturbations (dominant part in Eq. 262). On the other hand, frictional dissipation is the main process that destroys kinetic energy. A few remarks are appropriate here. If we forget for the time being about the approximative expressions given for an ideal gas, e_{APE}, and its evolution equation are really very general for a stably stratified one-component system. For example consider a simple solid body. We know from experience that even if a differential heating is applied, the only process that may result is heat diffusion (diabatic process), but certainly no kinetic energy can be gained. Indeed, in case of a solid body the function $q_0(s)$ can chosen as the internal energy u plus the constant potential energy, because $T = T(s)$, therefore we can define $T_0(s) = T$. In case of a simple incompressible fluid, we can also identify $T_0(s) = T$ and $q_0(s)$ can be chosen to equal to the internal energy, but the potential energy may not be constant, because of fluctuations of surface height, and could therefore provide the available potential energy.

To derive a slightly different formulation that has been derived by Lorenz (1955), equation has to be integrated over the systems volume

$$\int_\tau \frac{de_{ape}}{dt} d\tau = \int_\tau \left\{ -p \frac{d\alpha}{dt} + p_R \alpha \nabla \cdot \mathbf{v} - \frac{\alpha}{\alpha_R} \mathbf{v} \cdot \nabla \phi + \frac{d\phi}{dt} + \frac{T - T_o(s)}{T} q \right\} d\tau \quad , \tag{263}$$

where $d\tau = \rho dx dy dz$ is a mass element. Using

$$p \frac{d\alpha}{dt} = \frac{d}{dt}(p\alpha) - \alpha \frac{dp}{dt}$$

where $\alpha = 1/\rho$. With $P = R/c_v U$ (see exercise 1!), it follows

$$\int_\tau \frac{de_{ape}}{dt} d\tau = \int_\tau \left\{ \frac{RT}{p} \omega + \frac{T - T_o(s)}{T} q \right\} d\tau \quad , \tag{264}$$

where the orographic term from the exercise has been ignored. This is the integral available potential energy balance for a hydrostatic atmosphere that was first derived by Lorenz (1955). Apart from the diabatic production term, which can be reformulated for an idealized gas to get a formulation identical to Lorenz, this form

is highlighting the conversion term to kinetic energy $-\omega RT/p$, which states that energy is converted into kinetic energy by *rising of warm air and sinking of cold air*. This makes the conversion term positive on average. This process is lowering the *centre of mass* of the atmosphere, thus releasing kinetic energy. In baroclinic waves, warm air is typically moving northward, and cold air southward. This together with the tendency for motions to be adiabatic, thus following lines of contant potential temperature (see Fig. 3), means that the warm air moving to the north tends to rise and the cold air moving to the south tends to sink, thus providing the conditions for conversion of available potential energy into kinetic energy.
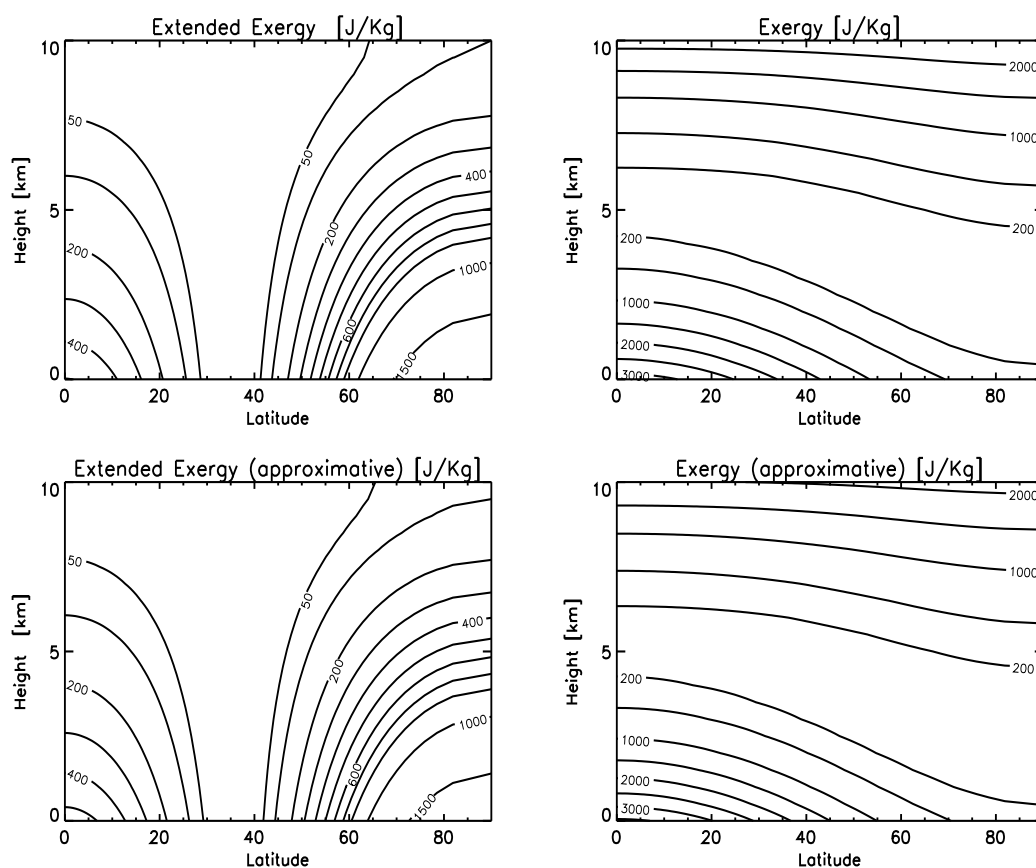

Figure 53: Distribution of classical Exergy and extended Exergy (local available potential energy). Units are J/kg.

It is further possible to decompose the kinetic and available potential energy in their zonal mean and eddy components (see Section 9.1), and to derive their evolution equations. If we identify the global volume averages of k and e_{ape} and their mean and eddy components as E_K, E'_K and E_P, E'_p then the resulting global
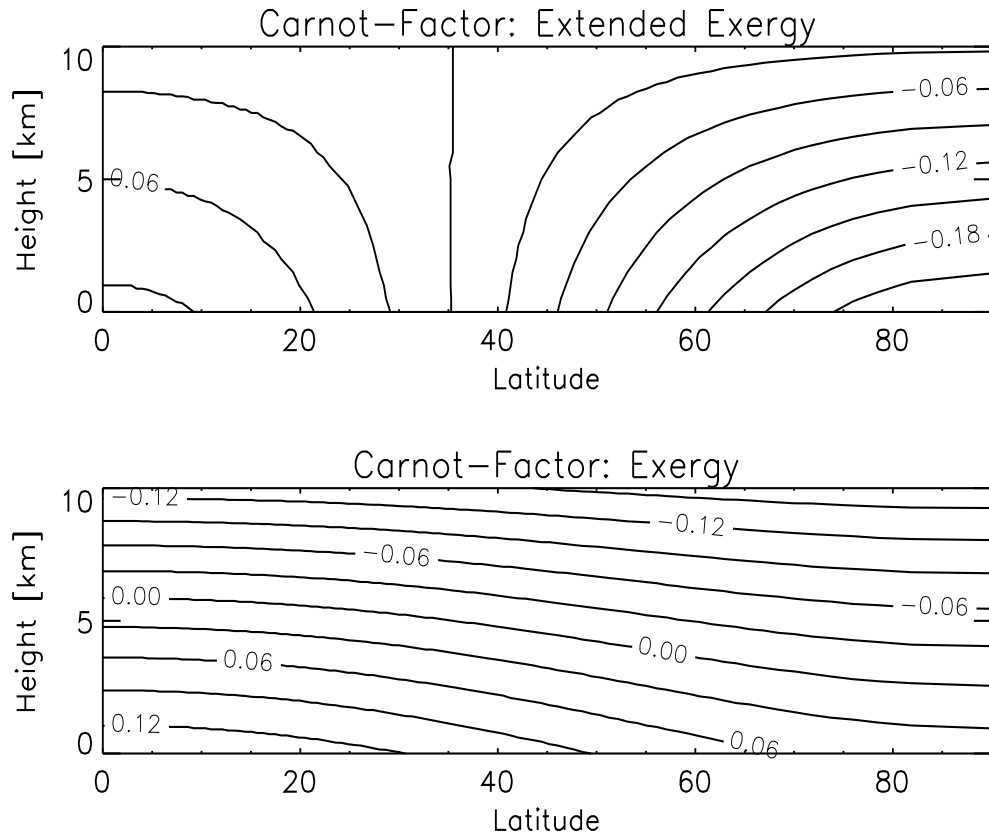
Figure 54: Distribution of Carnot factors of Exergy and extended Exergy (local available potential energy).

energy cycle according to Peixoto and Oort (1983) is shown in Fig. 55. As can be seen most production going into E_p, but also a considerable part into E'_P. These are related to heating in equatorial regions and cooling in polar regions for E_P, and diabatic heat release in cyclones in case of E'_P. The energy is then transformed into eddy kinetic energy, E'_K, from where most of the dissipation occurs. A smaller part is transferred into mean kinetic energy E_K, and eventually dissipated and also transformed back into E_P.

For the ocean we have a much simpler picture, mainly driver by atmospheric winds and friction:

Exercises

1. Show that for a hydrostatic atmosphere the volume integral of potential energy is proportional to the volume integral of internal energy, i.e. $P = R/c_v U$.
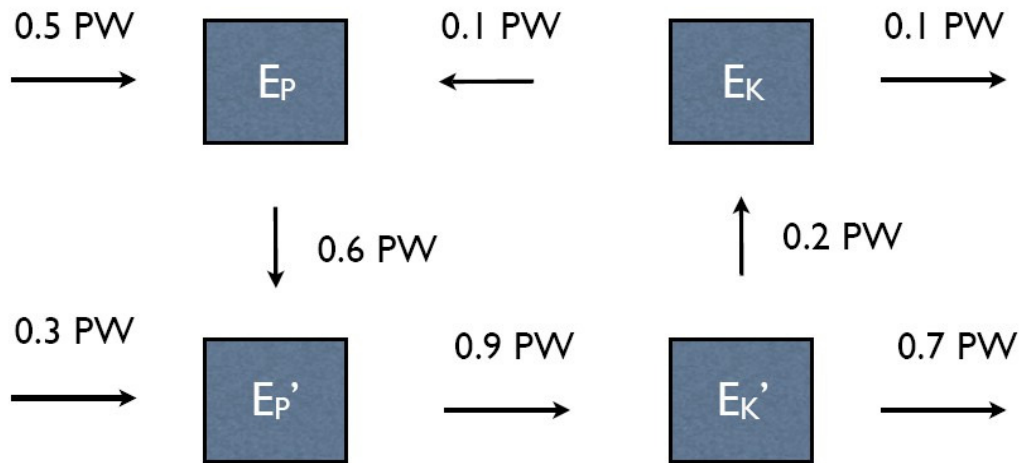
Figure 55: The global atmospheric energy cycle for the global integrals of mean and eddy available potential and kinetic energies (E_P, E'_P, E_K, E'_K), respecitvely.

(Hint: Use a partial integration!)

$$P = \int\int_A \int_{Z_s}^{\infty} \rho g z \, dz \, dx \, dy = \frac{R}{g} \int\int_A \int_0^{p_s} T \, dp \, dx \, dy + \int\int_A \frac{\Phi_s p_s}{g} \, dx \, dy$$

2. Show that

$$T - T_0(s) \approx \left(\frac{g^2}{c_p N_R^2} \frac{\Delta \theta}{\theta_R} + T_R \frac{R}{c_p} \frac{\Delta p}{p_R} \right)$$

in first order by developing $T(s, p)$ and $T_0(s)$ in taylor series around a reference state s_R, α_R. Hint: Note that $\partial T/\partial s = T/c_p$, $\partial T/\partial p = \alpha/c_p$ and

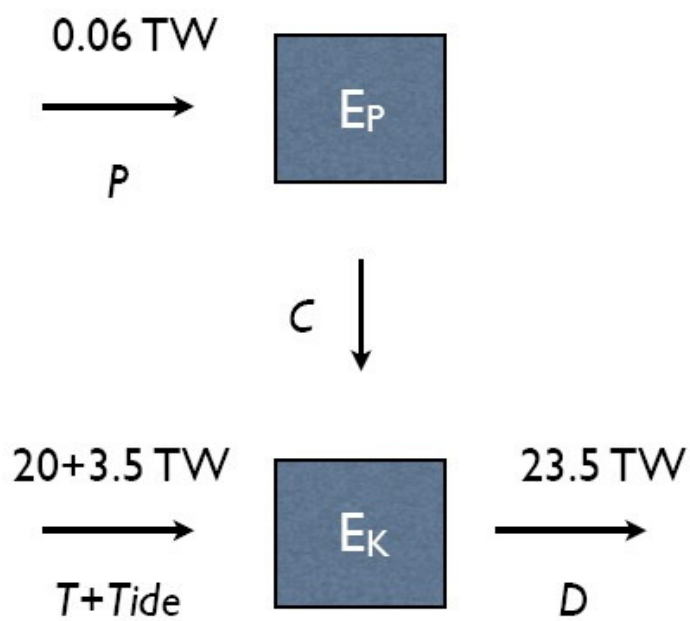$$\frac{dT_0}{ds}(s_R) = \frac{dT_R}{dz} \left(\frac{ds_R}{dz} \right)^{-1}$$

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Figure 56: The global oceanic energy cycle therefore the global integrals of available potential and kinetic energies.